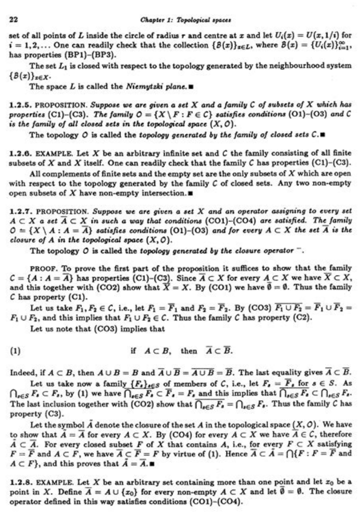

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set of all points of L inside the circle of radius r and centre at x and let $U_i(x) = U(x, 1/i)$ for $i = 1, 2, \ldots$ One can readily check that the collection $\{\mathcal{B}(x)\}_{x \in L}$, where $\mathcal{B}(x) = \{U_i(x)\}_{i=1}^{\infty}$, has properties (BP1)–(BP3).

The set L_1 is closed with respect to the topology generated by the neighbourhood system $\{\mathcal{B}(x)\}_{x \in X}$.

The space L is called the *Niemytzki plane.*■

1.2.5. PROPOSITION. *Suppose we are given a set X and a family $\mathcal{C}$ of subsets of X which has properties* (C1)–(C3). *The family $\mathcal{O} = \{X \setminus F : F \in \mathcal{C}\}$ satisfies conditions* (O1)–(O3) *and $\mathcal{C}$ is the family of all closed sets in the topological space $(X, \mathcal{O})$.*

The topology $\mathcal{O}$ is called the *topology generated by the family of closed sets $\mathcal{C}$.*■

1.2.6. EXAMPLE. Let X be an arbitrary infinite set and $\mathcal{C}$ the family consisting of all finite subsets of X and X itself. One can readily check that the family $\mathcal{C}$ has properties (C1)–(C3).

All complements of finite sets and the empty set are the only subsets of X which are open with respect to the topology generated by the family $\mathcal{C}$ of closed sets. Any two non-empty open subsets of X have non-empty intersection.■

1.2.7. PROPOSITION. *Suppose we are given a set X and an operator assigning to every set $A \subset X$ a set $\overline{A} \subset X$ in such a way that conditions* (CO1)–(CO4) *are satisfied. The family $\mathcal{O} = \{X \setminus A : A = \overline{A}\}$ satisfies conditions* (O1)–(O3) *and for every $A \subset X$ the set $\overline{A}$ is the closure of A in the topological space $(X, \mathcal{O})$.*

The topology $\mathcal{O}$ is called the *topology generated by the closure operator* $^-$.

PROOF. To prove the first part of the proposition it suffices to show that the family $\mathcal{C} = \{A : A = \overline{A}\}$ has properties (C1)–(C3). Since $\overline{A} \subset X$ for every $A \subset X$ we have $\overline{X} \subset X$, and this together with (CO2) show that $\overline{X} = X$. By (CO1) we have $\overline{\emptyset} = \emptyset$. Thus the family $\mathcal{C}$ has property (C1).

Let us take $F_1, F_2 \in \mathcal{C}$, i.e., let $F_1 = \overline{F}_1$ and $F_2 = \overline{F}_2$. By (CO3) $\overline{F_1 \cup F_2} = \overline{F}_1 \cup \overline{F}_2 = F_1 \cup F_2$, and this implies that $F_1 \cup F_2 \in \mathcal{C}$. Thus the family $\mathcal{C}$ has property (C2).

Let us note that (CO3) implies that

$$\text{(1)} \qquad \text{if} \quad A \subset B, \quad \text{then} \quad \overline{A} \subset \overline{B}.$$

Indeed, if $A \subset B$, then $A \cup B = B$ and $\overline{A} \cup \overline{B} = \overline{A \cup B} = \overline{B}$. The last equality gives $\overline{A} \subset \overline{B}$.

Let us take now a family $\{F_s\}_{s \in S}$ of members of $\mathcal{C}$, i.e., let $F_s = \overline{F}_s$ for $s \in S$. As $\bigcap_{s \in S} F_s \subset F_s$, by (1) we have $\overline{\bigcap_{s \in S} F_s} \subset \overline{F}_s = F_s$ and this implies that $\overline{\bigcap_{s \in S} F_s} \subset \bigcap_{s \in S} F_s$. The last inclusion together with (CO2) show that $\overline{\bigcap_{s \in S} F_s} = \bigcap_{s \in S} F_s$. Thus the family $\mathcal{C}$ has property (C3).

Let the symbol $\tilde{A}$ denote the closure of the set A in the topological space $(X, \mathcal{O})$. We have to show that $\tilde{A} = \overline{A}$ for every $A \subset X$. By (CO4) for every $A \subset X$ we have $\overline{A} \in \mathcal{C}$, therefore $\tilde{A} \subset \overline{A}$. For every closed subset F of X that contains A, i.e., for every $F \subset X$ satisfying $F = \overline{F}$ and $A \subset F$, we have $\overline{A} \subset \overline{F} = F$ by virtue of (1). Hence $\overline{A} \subset \tilde{A} = \bigcap\{F : F = \overline{F} \text{ and } A \subset F\}$, and this proves that $\tilde{A} = \overline{A}$.■

1.2.8. EXAMPLE. Let X be an arbitrary set containing more than one point and let x_0 be a point in X. Define $\overline{A} = A \cup \{x_0\}$ for every non-empty $A \subset X$ and let $\overline{\emptyset} = \emptyset$. The closure operator defined in this way satisfies conditions (CO1)–(CO4).